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ON WAVE SOLUTIONS OF THE HEAT-CONDUCTION EQUATION

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Abstract—Some examples are presented of wave equations of a parabolic-type heat-conduction equation. Their incorrectness is demonstrated. Further the derivation is presented of the hyperbolic heat-conduction equation wherein the velocity of isotherm displacement along the normal is used as an experimental parameter.

The above examples are analysed on the basis of the heat-conduction equations of a hyperbolic type and their correctness is proved following Hadamard.

NOMENCLATURE

- T, temperature;
- t, time;
- X, spatial coordinate;
- k, thermal conductivity;
- γ, heat capacity.

1. ISOTHERMS FOR WAVE SOLUTIONS

FOR A one-dimensional case the classical heat-conduction equation is of the form

$$\gamma \frac{\partial T}{\partial t} = \frac{\partial}{\partial X}\left(k \frac{\partial T}{\partial X}\right). \tag{1.0}$$

If γ and k are constant, thermal conductivity may be included into a spatial coordinate with the aid of the transform $x = \sqrt{(\gamma/k)}X$

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}. \tag{1.1}$$

Equation (1.1) may be satisfied by two wave solutions. A rightward travelling wave is defined by the formula

$$T_1(x,t) = A\exp[-c(x-ct)] \tag{1.2}$$

and the inverse wave is of the form

$$T_2(x,t) = A\exp[c(x+ct)]. \tag{1.3}$$

If further we pass over to the imaginary numbers $c = in$, $A = 2iA_n$, then the difference $T = T_2 - T_1$ will be expressed by the following formula

$$T(x,t) = A_n \exp[-n^2 t]\sin nx. \tag{1.4}$$

At $t_0 \leqslant t \leqslant 0$, $0 \leqslant x \leqslant \pi$ formula (1.4) may be treated as a solution of the temperature history problem of a heated body from its state at the given moment.

It is not difficult to demonstrate that at $t_0 \leqslant t \leqslant 0$ the problem under consideration is incorrect according to Hadamard [1]. Really, let us evaluate a norm of close initial data

$$\| A_n \sin nx - 0 \| = \max_{\{0 \leqslant x \leqslant \pi\}} |A_n \sin nx| \leqslant A_n. \tag{1.5}$$

Following Hadamard, let A_n be equal to $e^{-\sqrt{n}}$. It should be borne in mind that at $n \to \infty$ expression (1.5) vanishes.

Next, consider a norm of solution differences satisfying the initial data

$$\|\exp[-(\sqrt{n}) - n^2 t]\sin nx - 0\| = \max_{\left\{\substack{0 \leqslant x \leqslant \pi \\ t_0 \leqslant t \leqslant 0}\right\}} |\exp[-(\sqrt{n}) - n^2 t]\sin nx| \leqslant \frac{\exp[n^2|t_0|]}{\exp[\sqrt{n}]}. \tag{1.6}$$

Hence it follows that at $n \to \infty$ the norm of solution differences becomes infinitely large at finite t_0 when the difference between the initial data tends to zero. Therefore, the proposed problem appears to be incorrect according to Hadamard.

In addition to incorrectness, solution (1.4) possesses one more peculiar feature.

Let us determine a line of equal temperatures $T(X,t) = \text{const}$. The following equality is valid for it

$$\frac{\partial T}{\partial t} + \frac{\mathrm{d}X}{\mathrm{d}t}\frac{\partial T}{\partial X} = 0. \tag{1.7}$$

Designate the velocity

$$C(X,t) = \frac{\mathrm{d}X}{\mathrm{d}t}.$$

In the considered one-dimensional case C coincides with the velocity of an isotherm which, in its turn, is determined as a ratio between infinitesimal increment of the normal external to an isotherm and infinitesimal time interval.

It follows from (1.7) that

$$C(X,t) = -\frac{\dfrac{\partial T}{\partial t}}{\dfrac{\partial T}{\partial X}}. \tag{1.8}$$

Now, for solution of (1.4) formula (1.8) takes the form

$$C = \sqrt{\left(\frac{k}{\gamma}\right)} n \tan\left[nX \cdot \sqrt{\left(\frac{\gamma}{k}\right)}\right] = \sqrt{\left(\frac{k}{\gamma}\right)} n \tan nx. \tag{1.9}$$

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An analysis of formula (1.9) reveals that there are points where velocity C vanishes or becomes infinite.

The marked peculiarities should make investigators be careful when employing the heat-conduction equation (1.1).

Employing a heat-conduction equation in the form (1.1), we may relate the velocity C to heat capacity and thermal conductivity as

$$C(X,t) = -\frac{\frac{1}{\gamma}\frac{\partial}{\partial X}\left(K\frac{\partial T}{\partial X}\right)}{\frac{\partial T}{\partial X}}.$$

In a particular case this formula permits us to consider $c = c(T)$ by analogy with the classical heat-conduction theory, i.e. the relation with the coordinates X and t may be implicit through temperature.

2. DERIVATION OF THE HYPERBOLIC HEAT-CONDUCTION EQUATION

Rewrite equation (1.7) as

$$\frac{\partial T}{\partial t} + C(X,t)\frac{\partial T}{\partial X} = 0. \tag{2.1}$$

Differentiation of equation (2.1) in succession with respect to X and t followed by elimination of the mixed derivative $\partial^2 T/\partial X\partial t$ from the two equalities obtained results in

$$\frac{\partial^2 T}{\partial t^2} + \left(\frac{\partial c}{\partial t} - C\frac{\partial c}{\partial X}\right)\frac{\partial T}{\partial X} = C^2\frac{\partial^2 T}{\partial X^2}. \tag{2.2}$$

By using (1.8), we eliminate $\partial T/\partial X$ and rewrite the above equality as

$$\frac{1}{c^2}\frac{\partial^2 T}{\partial t^2} + \frac{1}{c^2}\left[\frac{\partial c}{\partial X} - \frac{1}{c}\frac{\partial c}{\partial t}\right]\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial X^2}. \tag{2.3}$$

Equation (2.3) turns into (1.1) if $c \to \infty$ and

$$\frac{1}{c^2}\left[\frac{\partial c}{\partial X} - \frac{1}{c}\frac{\partial c}{\partial t}\right] \to \frac{\gamma}{K}.$$

The condition $c \to \infty$ does not however imply that the classical heat-conduction equation (1.1) describes only the process when isotherms move with infinite velocity. Really, in wave solutions (1.2) and (1.3) the isotherms have finite velocities.

In case of the constant isotherm velocity, equation (2.3) is simplified to the form of the classical wave equation

$$\frac{\partial^2 T}{\partial t^2} = C^2\frac{\partial^2 T}{\partial X^2}. \tag{2.4}$$

Based on the theory of Riemannian manifolds this equation was obtained by A. S. Predvoditelev [2].

Let $c = c(X)$, then equation (2.3) acquires the form

$$\frac{1}{c^2}\frac{\partial^2 T}{\partial t^2} + \frac{1}{c^2}\frac{\mathrm{d}c}{\mathrm{d}X}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial X^2}. \tag{2.5}$$

If $(1/c^2)(\mathrm{d}c/\mathrm{d}X) = (\gamma/K)$, where γ and K are heat capacity and thermal conductivity, respectively, then

$$\left(\frac{\gamma}{K}X + \text{const}\right)^2\frac{\partial^2 T}{\partial t^2} + \frac{\gamma}{K}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial X^2}.$$

Divide both parts of the last relationship by γ/k

$$\left[\sqrt{\left(\frac{\gamma}{k}\right)}X + \text{const}\sqrt{\left(\frac{k}{\gamma}\right)}\right]^2\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \frac{k}{\gamma}\frac{\partial^2 T}{\partial X^2}.$$

Since k/γ is the constant value, thermal conductivity may be included into a spatial coordinate with the aid of the particular transform $x = \sqrt{(\gamma/k)}X$; besides, in some cases the cofactor at $\partial^2 T/\partial t^2$ may be approximately assumed equal to the constant parameter λ. Then, finally

$$\lambda\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}. \tag{2.6}$$

In unsteady heat-transfer problems the isotherm velocity is likely to depend on time. Then equation (2.3) becomes

$$\frac{1}{c^2}\frac{\partial^2 T}{\partial t^2} - \frac{1}{c^3}\frac{\mathrm{d}c}{\mathrm{d}t}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial X^2}. \tag{2.7}$$

In a particular case equation (2.7) may turn into (2.6)

3. SOLUTION OF SOME INCORRECTLY STATED HEAT-CONDUCTION PROBLEMS

The reported disadvantages of heat conduction equation (1.1) make new equation (2.6) preferable.

Moreover, from the point of view of the existing methods of solving incorrectly stated problems

$$\left(\lambda\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)$$

may be considered as an operator close to the initial one $[(\partial/\partial t) - (\partial^2/\partial x^2)]$ at small λ.

By using equation (2.6) it is easy to obtain solutions for direct and inverse waves in the form

$$T_1(x,t) = A\exp\left[-\frac{c}{1-\lambda c^2}(x-ct)\right];$$

$$T_2(x,t) = A\exp\left[\frac{c}{1-\lambda c^2}(x+ct)\right].$$

Assuming $c = in$, $A = 2iA_n$ and composing $T_2 - T_1$ yield

$$T(x,t) = A_n\exp\left[-\frac{n^2 t}{1+\lambda n^2}\right]\sin\frac{nx}{1+\lambda n^2}. \tag{3.1}$$

It is evident that at $\lambda \to 0$ solution (3.1) converges to solution (1.4).

Next, we shall demonstrate that close solutions correspond to close initial data

$$T_1 = \exp[-\sqrt{n}]\sin\frac{nx}{1+\lambda n^2}$$

and $T_2 \equiv 0$.

Really

$$\|T_1 - T_2\| = \max_{\{0 \leqslant x \leqslant \pi\}}|\exp[-\sqrt{n}]\sin\frac{nx}{1+\lambda n^2} - 0| \leqslant \exp[-\sqrt{n}] \tag{3.2}$$

$$\|T_1(x,t) - T_2(x,t)\| = \max_{\substack{0 \leqslant x \leqslant \pi \\ t_0 \leqslant t \leqslant 0}}\left|\exp\left[-(\sqrt{n}) - \frac{n^2 t}{1+\lambda n^2}\right]\sin\frac{nx}{1+\lambda n^2}\right| \leqslant \exp\left[-(\sqrt{n}) - \frac{n^2 t_0}{1+\lambda n^2}\right]. \tag{3.3}$$

Hence it follows that at $\lambda \neq 0$ and $n \to \infty$ right-hand sides of (3.2) and (3.3) vanish that implies disappearance of instability.

Note, that solution (3.1) may be obtained by using a non-homogeneous heat-conduction equation of the type

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} - \lambda\left(\frac{n^2}{1+\lambda n^2}\right) \times \exp\left[-\frac{n^2 t}{1-\lambda n^2}\right]\sin\frac{nx}{1+\lambda n^2}. \quad (3.4)$$

Here the function

$$f(x, t, \lambda) = -\lambda\left(\frac{n^2}{1+\lambda n^2}\right)^2 \exp\left[-\frac{n^2 t}{1+\lambda n^2}\right]\sin\frac{nx}{1+\lambda n^2}$$

characterizes generation and absorption of heat. Addition to equation (1.1) of a non-homogeneous term vanishing at $\lambda \to 0$ and corresponding to $\lambda(\partial^2 T/\partial t^2)$ in equation (2.6) means that from the point of view of general approach to solution of incorrectly stated problems the initial operator is replaced by a close one. On the other hand, presence of positive and negative heat sources arranged in a special way explains a possibility for non-damping solutions to exist.

Calculate $c(x, t)$ for (3.1)

$$c = \sqrt{\left(\frac{k}{\gamma}\right)} n \tan\frac{nx}{1+\lambda n^2}. \quad (3.5)$$

If for solution (1.4) obtained from equation (1.1) there are points where $c = 0$ or $c = \infty$ then for solution (3.1), as it is seen from (3.5), this peculiarity may be eliminated by a correct choice of λ. It means that heating through the substance from a boundary into the region $0 < x < \pi$ will proceed with a finite rate different from zero.

As another example of unsteady-state solution consider a temperature wave at $x > 0$:

$$T(x, t) = A_n \exp\left[\sqrt{\left(\frac{n}{2}\right)} x\right] \cos\left[\sqrt{\left(\frac{n}{2}\right)} x + nt\right]. \quad (3.6)$$

Function $T(x, t)$ defined by (3.6) satisfies heat-conduction equation (1.1) and the boundary condition [3]

$$T(0, t) = A_n \cos nt.$$

We shall prove incorrectness of the problem. Take A_n equal to $1/n$ and compare the boundary conditions $T_1(0, t) = 1/n \cos nt$ and $T_2(0, t) \equiv 0$.

$$\| T_1(0, t) - T_2(0, t) \| = \max_{\{t \geqslant 0\}} \left|\frac{1}{n}\cos nt\right| = \frac{1}{n}. \quad (3.7)$$

Next, compare at some limited region $0 \leqslant x \leqslant x_0$ the solutions $T_1 = 1/n \exp[\sqrt{(n/2)}x] \cos[\sqrt{(n/2)}x + nt]$ and $T_2 \equiv 0$ corresponding to the boundary conditions

$$\| T_1(x, t) - T_2(x, t) \| = \max_{\left\{\substack{0 \leqslant x \leqslant x_0 \\ t \geqslant 0}\right\}} \left|\frac{1}{n}\exp\left[\sqrt{\left(\frac{n}{2}\right)} x\right] \cos\left[\sqrt{\left(\frac{n}{2}\right)} x + nt\right]\right| \leqslant \frac{\exp\left[\sqrt{\left(\frac{n}{2}\right)} x_0\right]}{n}. \quad (3.8)$$

It can be seen here that at $n \to \infty$ expression (3.7) vanishes and (3.8) tends to infinity, i.e. infinitely close solutions correspond to infinitely close boundary conditions. Turning again to solution (2.6) it is easy to get a solution similar to the temperature wave (3.6) as

$$T(x, t) = A_n \exp\left\{\left[\frac{n}{2}(\sqrt{\lambda^2 n^2 + 1} - \lambda n)\right]^{\frac{1}{2}} x\right\} \times \cos\left\{\left[\frac{n}{2}(\sqrt{\lambda^2 n^2 + 1} - \lambda n)\right]^{\frac{1}{2}} x + nt\right\}. \quad (3.9)$$

Now close solutions

$$T_1 = \frac{1}{n}\exp\left\{\left[\frac{n}{2}(\sqrt{\lambda^2 n^2 + 1} - \lambda n)\right]^{\frac{1}{2}} x\right\} \times \cos\left\{\left[\frac{n}{2}(\sqrt{\lambda^2 n^2 + 1} - \lambda n)\right]^{\frac{1}{2}} x + nt\right\}$$

and $T_2 \equiv 0$ correspond to close boundary conditions $T_1(0, t) = 1/n \cos nt$ and $T_2(0, t) \equiv 0$. Indeed, consider the norm of the difference

$$\| T_1(x, t) - T_2(x, t) \| = \max_{\left\{\substack{0 \leqslant x \leqslant x_0 \\ t \geqslant 0}\right\}} \left|\frac{1}{n}\exp\left\{\left[\frac{n}{2}(\sqrt{\lambda^2 n^2 + 1} - \lambda n)\right]^{\frac{1}{2}} x\right\} \times \cos\left\{\left[\frac{n}{2}(\sqrt{\lambda^2 n^2 + 1} - \lambda n)\right]^{\frac{1}{2}} x + nt\right\}\right| \leqslant \frac{1}{n}\exp\left\{\left[\frac{n}{2}(\sqrt{\lambda^2 n^2 + 1} - \lambda n)\right]^{\frac{1}{2}} x_0\right\}. \quad (3.10)$$

Since

$$\lim_{n \to \infty} \frac{n}{2}[\sqrt{(\lambda^2 n^2 + 1)} - \lambda n] x_0 = \frac{x_0}{4\lambda}$$

then at each fixed x_0 and $\lambda \neq 0$ a RHS of inequality (3.10) tends to zero at $n \to \infty$ that testifies to stability of the presented solutions based on the initial data.

The examples considered may suggest an idea of employing hyperbolic heat-conduction equation (2.3) for description of wave propagation of temperature since expression (2.3) does away with many unpleasant peculiarities of solutions when classical equation (1.1) is used.

The equation of hyperbolic type is preferable for describing thermal effects due to the remarkable fact that many solutions of heat-conduction equation and even a fundamental one may be represented as combinations of temperature waves of the form

$$\exp\left[\pm\sqrt{\left(\frac{n}{2}\right)} x\right] \cos\left[\pm\sqrt{\left(\frac{n}{2}\right)} x + nt + \varepsilon\right], \quad [4].$$

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SUR DES SOLUTIONS ONDULATOIRES DE L'EQUATION DE LA CHALEUR

Résumé—On donne quelques exemples de solutions ondulatoires de l'équation de la chaleur de type parabolique et leur inexactitude est démontrée. La formulation est alors présentée de l'équation de la chaleur hyperbolique dans laquelle la vitesse de déplacement d'une isotherme suivant la normale est utilisée comme paramètre expérimental.

Les exemples précédents sont analysés sur la base de l'équation de la chaleur de type hyperbolique et leur exactitude est montrée en s'appuyant sur Hadamard.

ÜBER WELLENLÖSUNGEN DER WÄRMELEITUNGSGLEICHUNG

Zusammenfassung—Es werden einige Beispiele angegeben für Wellengleichungen vom parabolischen Typ. Ihre Ungenauigkeit ist gezeigt. Weiterhin wird die Ableitung der hyperbolischen Wärmeleitungsgleichung wiedergegeben, wobei die Geschwindigkeit der Isothermenausbreitung längs der Normalen als ein experimenteller Parameter eingeführt ist. Die obigen Beispiele werden analysiert aufgrund der Wärmeleitungsgleichungen vom hyperbolischen Typ, und ihre Übereinstimmung wird nach Hadamard nachgewiesen.

О ВОЛНОВЫХ РЕШЕНИЯХ УРАВНЕНИЯ ТЕПЛОПРОВОДНОСТИ

Аннотация — Приведены примеры волновых решений уравнения теплопроводности параболического типа. Показывается их некорректность. Далее дается вывод гиперболического уравнения теплопроводности, в котором в качестве экспериментального параметра используется скорость перемещения изотермы по нормали.

Указанные примеры разбираются на основе уравнения теплопроводности гиперболического типа и доказывается их корректность по Адамару.